

Note di Matematica

ISSN 1123-2536, e-ISSN 1590-0932

Note Mat. **37** (2017) suppl. 1, 93–102.

doi:10.1285/i15900932v37suppl1p93

Local controllability of trident snake robot based on sub-Riemannian extremals

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Received: 29-07-2016; accepted: 04-11-2016.

Abstract. To solve trident snake robot local controllability by differential geometry tools, we construct a privileged system of coordinates with respect to the distribution given by Pfaff system based on local nonholonomic conditions and, furthermore, we construct a nilpotent approximation of the transformed distribution with respect to the given filtration. We compute normal extremals of sub-Riemannian structure, where the Hamiltonian point of view was used. We demonstrated that the extremals of sub-Riemannian structure based on this distribution play the similar role as classical periodic inputs in control theory with respect of our mechanism.

Keywords: local controllability, nonholonomic mechanics, planar mechanisms, sub-Riemannian geometry, differential geometry.

MSC 2010 classification: primary 53A17, secondary 70H05, 70Q05

1 Introduction

Originally, the general trident snake robot has been introduced in [5]. It is a planar robot with a body in the shape of a triangle and with three legs consisting of ℓ links. Then, its simplest non-trivial version (see figure 1), corresponding to $\ell = 1$, has been mainly discussed, see e.g. [6],[4]. Local controllability of such robot is given by the appropriate Pfaff system of ODEs. The solution with respect to

$$\dot{q} = (\dot{x}, \dot{y}, \dot{\theta}, \dot{\Phi}_1, \dot{\Phi}_2, \dot{\Phi}_3)$$

gives a control system $\dot{q} = Gu$. In the case length one links the control matrix G is a 6×3 matrix spanned by vector fields g_1, g_2, g_3 , where

$$\begin{aligned} g_1 &= \cos \theta \partial_x - \sin \theta \partial_y + \sin \Phi_1 \partial_{\Phi_1} + \sin(\Phi_2 + \frac{2\pi}{3}) \partial_{\Phi_2} + \sin(\Phi_3 + \frac{4\pi}{3}) \partial_{\Phi_3}, \\ g_2 &= \sin \theta \partial_x + \cos \theta \partial_y - \cos \Phi_1 \partial_{\Phi_1} - \cos(\Phi_2 + \frac{2\pi}{3}) \partial_{\Phi_2} - \cos(\Phi_3 + \frac{4\pi}{3}) \partial_{\Phi_3}, \\ g_3 &= \partial_\theta - (1 + \cos \Phi_1) \partial_{\Phi_1} - (1 + \cos \Phi_2) \partial_{\Phi_2} - (1 + \cos \Phi_3) \partial_{\Phi_3} \end{aligned}$$

ⁱThis work was supported by a grant of the Czech Science Foundation (GAČR) no. 17-21360S.

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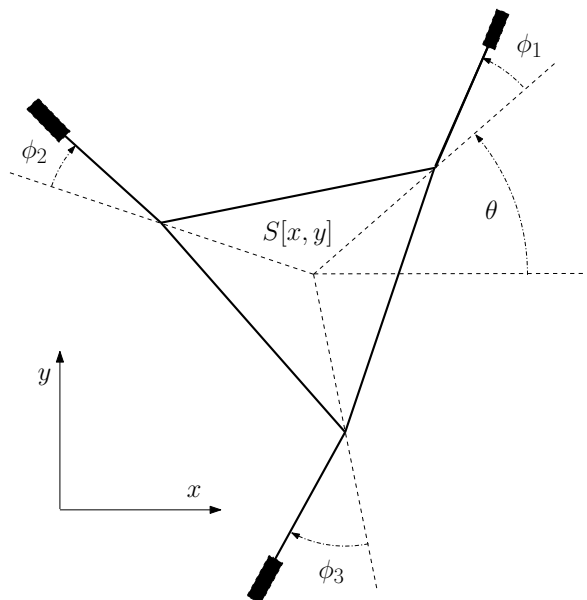


Figure 1. 1-link trident snake robot model

and $u : \mathbb{R} \rightarrow \mathbb{R}^3$ is the control of our system. It is easy to check that in regular points these vector fields define a (bracket generating) distribution H with growth vector $(3, 6)$. It means that in each regular point the vectors g_1, g_2, g_3 together with their Lie brackets span the whole tangent space. Consequently, the system is controllable by Chow–Rashevsky theorem.

The sub-Riemannian structure on manifold M is a generalization of Riemannian manifold. Given a distribution $H \subset TM$ equipped with metrics based on control u we can define so called Carnot–Carathéodory metric on M and sub-Riemannian Hamiltonian.

2 Nilpotent approximation

In order to simplify the trident snake robot control we construct a nilpotent approximation of the transformed distribution with respect to the given filtration. Note that all constructions are local in the neighborhood of 0. Following [3], we group together the monomial vector fields of the same weighted degree and thus we express $g_i, i = 1, 2, 3$ as a series

$$g_i = g_i^{(-1)} + g_i^{(0)} + g_i^{(1)} + \cdots,$$

where $g_i^{(s)}$ is a homogeneous vector field of order s . Then the following proposition holds, [7]. Set $X_i = g_i^{(-1)}$, $i = 1, 2, 3$. The family of vector fields (X_1, X_2, X_3) is a first order approximation of (g_1, g_2, g_3) at 0 and generates a nilpotent Lie algebra of step $r = 2$, i.e. all brackets of length greater than 2 are zero. In our case [3], we obtain the following vector fields:

$$\begin{aligned} X_1 &= \partial_x + \left(-\frac{y}{2}\right) \partial_a + \left(-\frac{y}{2} - d\right) \partial_b + \left(-\frac{x}{2}\right) \partial_c, \\ X_2 &= \partial_y + \left(\frac{x}{2}\right) \partial_a - \left(\frac{x}{2}\right) \partial_b + \left(\frac{y}{2} - d\right) \partial_c, \\ X_3 &= \partial_d. \end{aligned}$$

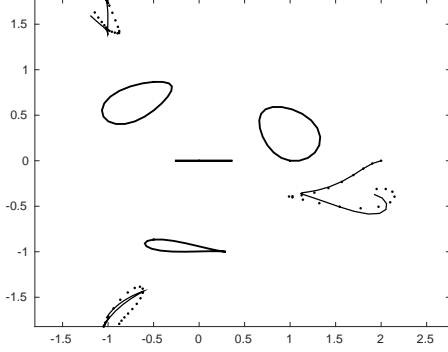
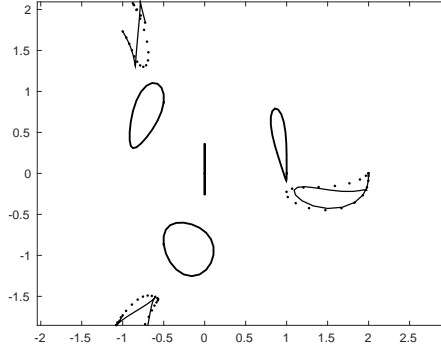
with respect to a new coordinates (x, y, d, a, b, c) . In particular, the family of vector fields (X_1, X_2, X_3) is the nilpotent approximation of (g_1, g_2, g_3) at 0 associated with the coordinates (x, y, d, a, b, c) . The remaining three vector fields are generated by Lie brackets of (X_1, X_2, X_3) . Note that due to linearity of the three latter coordinates of (X_1, X_2, X_3) , the coordinates of (X_4, X_5, X_6) must be constant. We get

$$\begin{aligned} X_4 &= \partial_a, \\ X_5 &= \partial_b, \\ X_6 &= \partial_c. \end{aligned}$$

Note that the vector fields $(X_1, X_2, X_3, X_4, X_5, X_6)$ in $(x, y, \theta, \Phi_1, \Phi_2, \Phi_3)$ coordinates are of the form

$$\begin{aligned} X_1 &= \partial_x + \theta \partial_{\Phi_1} - \left(-\frac{\sqrt{3}}{2} + \frac{\theta}{2}\right) \partial_{\Phi_2} - \left(-\frac{\sqrt{3}}{2} + \frac{\theta}{2}\right) \partial_{\Phi_3}, \\ X_2 &= \partial_y - \partial_{\Phi_1} + \left(-\frac{1}{2} + \frac{\sqrt{3}\theta}{2}\right) \partial_{\Phi_2} - \left(-\frac{1}{2} + \frac{\sqrt{3}\theta}{2}\right) \partial_{\Phi_3}, \\ X_3 &= \partial_\theta - 2\partial_{\Phi_1} - 2\partial_{\Phi_2} - 2\partial_{\Phi_3}, \\ X_4 &= \partial_{\Phi_1} + \partial_{\Phi_2} + \partial_{\Phi_3}, \\ X_5 &= -\frac{\sqrt{3}}{2} \partial_{\Phi_2} + \frac{\sqrt{3}}{2} \partial_{\Phi_3}, \\ X_6 &= -\partial_\theta + \frac{1}{2} \partial_{\Phi_2} + \frac{1}{2} \partial_{\Phi_3}. \end{aligned}$$

To show how nilpotent approximation affects on integral curves of the distributions and the resulting control we computed the Lie brackets of relevant vector fields. In Fig. 2, there is a comparison of the Lie bracket g_5 motions realized in $(x, y, \theta, \Phi_1, \Phi_2, \Phi_3)$ coordinates (dotted line) and in nilpotent approximation. Fig. 3 show the comparison of g_6 motions. The following figures show the trajectories of the root center point, vertices and wheels when a particular Lie bracket motion is realized.

Figure 2. g_5 motionFigure 3. g_6 motion

3 sub-Riemannian Hamiltonian

Note that the functions in $C^\infty(M)$ are in one-to-one correspondence with functions in $C^\infty(T^*M)$ that are constant on fibers:

$$C^\infty(M) \cong C_{const}^\infty(T^*M) = \{\pi^*\alpha \mid \alpha \in C^\infty(M)\} \subset C^\infty(T^*M),$$

where $\pi : T^*M \rightarrow M$ denotes the canonical projection. In what follows, with no abuse of notation, we often identify the function $\pi^*\alpha \in C^\infty(T^*M)$ with the function $\alpha \in C^\infty(M)$. In a similar way, smooth vector fields on M are in a one-to-one correspondence with functions in $C^\infty(T^*M)$ that are linear on fibers via the map $Y \mapsto a_Y$, where $a_Y(\lambda) := \langle \lambda, Y(q) \rangle$ and $q = \pi(\lambda)$, i.e.

$$Vec(M) \cong C_{lin}^\infty(T^*M) = \{a_Y \mid Y \in Vec(M)\} \subset C^\infty(T^*M).$$

Our mechanism can be understood as a model of sub-Riemannian structure of constant rank, i.e. the triple $(M, D, \langle \cdot, \cdot \rangle)$, where D is a vector subbundle of TM locally generated by family of vector fields $\{f_1, \dots, f_m\}$ (in our case generated by X_1, X_2 and X_3) and $\langle \cdot, \cdot \rangle_q$ is a family of scalar products on D_q . Then the sub-Riemannian Hamiltonian is the function on T^*M defined as follows [1]

$$h : T^*M \rightarrow \mathbb{R}$$

$$h(\lambda) = \max_{u \in U_q} \left(\langle \lambda, f_u(q) \rangle - \frac{1}{2}|u|^2 \right), \quad q = \pi(\lambda),$$

where $f_u = \sum_i u_i f_i$. For every generating family $\{f_1, \dots, f_m\}$ of the sub-Riemannian structure, the sub-Riemannian Hamiltonian H can be rewritten [1] as follows

$$h(\lambda) = \frac{1}{2} \sum_i \langle \lambda, f_i(q) \rangle^2, \quad \lambda \in T_q^*M, \quad q = \pi(\lambda).$$

To simplify our equations below we can define the Poisson bracket as an binary operation $\{, \}$ on functions in $C^\infty(T^*M)$. In fact, we can define

$$\{a_X, a_Y\} := a_{[X, Y]} \quad (1)$$

and there exists a unique bilinear and skew-symmetric map

$$\{\cdot, \cdot\} : C^\infty(T^*M) \times C^\infty(T^*M) \rightarrow C^\infty(T^*M)$$

that extends Poisson bracket (1) on $C^\infty(T^*M)$, and that is a derivative (i.e. satisfies the Lipschitz rule) in each argument. Let (x, p) denote coordinates on T^*M , the formula for Poisson bracket of two functions $a, b \in C^\infty(T^*M)$ reads

$$\{a, b\} = \sum_{i=1}^n \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial p_i}$$

and the Hamiltonian vector field associated with the smooth function $a \in C^\infty(T^*M)$ is defines as the linear operation

$$\vec{a} : C^\infty(T^*M) \rightarrow C^\infty(T^*M), \quad \vec{a}(v) = \{a, v\}.$$

We can easily write the coordinate expression of \vec{a} for an arbitrary function $a \in C^\infty(T^*M)$ as

$$\vec{a} = \{a, \cdot\} = \sum_{i=1}^n \frac{\partial a}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial a}{\partial x_i} \frac{\partial}{\partial p_i}.$$

Let $\gamma : [0, T] \rightarrow M$ be an admissible curve with respect to controlling distribution which is lenght-minimizer, parametrized by constant speed. Let \bar{u} be the corresponding minimal control. Then there exists a Lipschitz curve $\lambda(t) \in T_{\gamma(t)}^*M$ such that

$$\dot{\lambda} = \sum_{i=1}^m \bar{u}_i(t) \vec{h}_i(\lambda(t)) = \vec{h}(t)(\lambda(t)) = \{h, \lambda\},$$

where $\vec{h} = \sum_{i=1}^m \bar{u}_i(t) \vec{h}_i$. On the controlling vector fields X_1, X_2 and X_3 and the general 1-form $\lambda \in T^*M$

$$\lambda = \lambda_x dx + \lambda_y dy + \lambda_d dd + \lambda_a da + \lambda_b db + \lambda_c dc$$

we define functions $h_i = \langle \lambda, X_i \rangle$

$$\begin{aligned} h_1(\lambda) &:= \lambda_x - \frac{y}{2} \lambda_a - \left(\frac{y}{2} + d \right) \lambda_b - \frac{x}{2} \lambda_c, \\ h_2(\lambda) &:= \lambda_y + \frac{x}{2} \lambda_a - \frac{x}{2} \lambda_b + \left(\frac{y}{2} - d \right) \lambda_c, \\ h_3(\lambda) &:= \lambda_d. \end{aligned}$$

In our case the sub-Riemannian Hamiltonian with respect to corresponding minimal control is then of the form

$$\begin{aligned} h(\lambda) &= u_1 h_1(\lambda) + u_2 h_2(\lambda) + u_3 h_3(\lambda) \\ &= u_1 \left(\lambda_x - \frac{y}{2} \lambda_a - \left(\frac{y}{2} + d \right) \lambda_b - \frac{x}{2} \lambda_c \right) \\ &\quad + u_2 \left(\lambda_y + \frac{x}{2} \lambda_a - \frac{x}{2} \lambda_b + \left(\frac{y}{2} - d \right) \lambda_c \right) + u_3 \lambda_d. \end{aligned}$$

In canonical coordinates (p, x) , the Hamiltonian vector field associated with h is expressed as follows

$$\vec{h} = \{h, \cdot\} = \sum_{i=1}^n \frac{\partial h}{\partial p_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial p_i}$$

and the Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda)$ is rewritten as

$$\dot{x}_i = \frac{\partial h}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial h}{\partial x_i}$$

i.e. in our case

$$\begin{aligned} \dot{x} &= \frac{\partial h}{\partial \lambda_x} = \frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial \lambda_x} = u_1 \\ \dot{y} &= \frac{\partial h}{\partial \lambda_y} = \frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial \lambda_y} = u_2 \\ \dot{d} &= \frac{\partial h}{\partial \lambda_d} = \frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial \lambda_d} = u_3 \end{aligned}$$

$$\begin{aligned} \dot{\lambda}_x &= -\frac{\partial h}{\partial x} = -\frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial x} = \frac{1}{2}(-\lambda_c u_1 + (\lambda_a - \lambda_b) u_2) \\ \dot{\lambda}_y &= -\frac{\partial h}{\partial y} = -\frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial y} = \frac{1}{2}(\lambda_a - \lambda_b) u_1 + (\lambda_c) u_2 \\ \dot{\lambda}_d &= -\frac{\partial h}{\partial d} = -\frac{\partial(u_1 h_1 + u_2 h_2 + u_3 h_3)}{\partial d} = \frac{1}{2}(-\lambda_b u_1 - (\lambda_c) u_2) \end{aligned}$$

$$\begin{aligned} \dot{\lambda}_a &= 0 & \dot{a} &= -\frac{y}{2} u_1 + \frac{x}{2} u_2 \\ \dot{\lambda}_b &= 0 & \dot{b} &= -\left(\frac{y}{2} + d\right) u_1 - \frac{x}{2} u_2 \\ \dot{\lambda}_c &= 0 & \dot{c} &= -\frac{x}{2} u_1 - \left(\frac{y}{2} - d\right) u_2 \end{aligned}$$

A function $a \in C^\infty(T^*M)$ is a constant of the motion of the Hamiltonian system associated with $h \in C^\infty(T^*M)$ if and only if $\{h, a\} = 0$. If $\gamma : [0, T] \rightarrow M$ is a length minimizer on sub-Riemannian manifold, associated with a control $u(\cdot)$, then due the Pontryagin maximum principle there exists $\lambda_0 \in T_{\gamma(0)}^*M$ such that defining

$$\lambda(t) = (P_{0,t}^{-1})^* \lambda_0, \quad \lambda(t) \in T_{\gamma(t)}^*M,$$

where $P_{0,t}$ is the flow of the nonautonomous vector field

$$X_{u(t)} = \sum_{i=1}^m u_i(t) X_i(\gamma(t))$$

and one of the following condition is satisfied

$$\begin{aligned} (N) \quad u_i(t) &= \langle \lambda(t), X_i(\gamma(t)) \rangle, \quad \forall i = 1, \dots, m, \\ (A) \quad 0 &= \langle \lambda(t), X_i(\gamma(t)) \rangle, \quad \forall i = 1, \dots, m. \end{aligned}$$

If $\lambda(t)$ satisfies (N) then it is called normal extremal (and $\gamma(t)$ a normal extremal trajectory). If $\lambda(t)$ satisfies (A) then it is called abnormal extremal (and $\gamma(t)$ a abnormal extremal trajectory).

4 Normal extremals

Following [1], every normal extremal is a solution of the Hamiltonian system $\dot{\lambda} = \vec{h}(\lambda(t))$, i.e. in our case of the system

$$\dot{\lambda} = \vec{h}(\lambda(t)) = \frac{1}{2}(h_1^2(\lambda) + h_2^2(\lambda) + h_3^2(\lambda)), \quad h_i(\lambda) = \langle \lambda, X_i(q) \rangle,$$

and let us introduce

$$\begin{aligned} h_4(\lambda) &:= \langle \lambda, X_4(q) \rangle, \\ h_5(\lambda) &:= \langle \lambda, X_5(q) \rangle, \\ h_6(\lambda) &:= \langle \lambda, X_6(q) \rangle. \end{aligned}$$

Since X_1, \dots, X_6 are linearly independent then $\{h_1, \dots, h_6\}$ defines a system coordinates on T^*M and thus we can consider λ to be parametrized by h_i

consequently, we are looking for a solution of the system $\dot{h}_i = \{h, h_i\}$, i.e.

$$\begin{aligned}\dot{h}_1 &= \{h, h_1(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_1\} = \{h_2, h_1\}h_2 + \{h_3, h_1\}h_3 = -h_4h_2 - h_5h_3 \\ \dot{h}_2 &= \{h, h_2(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_2\} = \{h_1, h_2\}h_1 + \{h_3, h_2\}h_3 = h_4h_1 - h_6h_3 \\ \dot{h}_3 &= \{h, h_3(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_3\} = \{h_1, h_3\}h_1 + \{h_2, h_3\}h_2 = h_5h_1 + h_6h_2 \\ \dot{h}_4 &= \vec{h}(h_4(t)) = \{h, h_4(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_4\} = 0 \\ \dot{h}_5 &= \vec{h}(h_5(t)) = \{h, h_5(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_5\} = 0 \\ \dot{h}_6 &= \vec{h}(h_6(t)) = \{h, h_6(t)\} = \{h_1^2 + h_2^2 + h_3^2, h_6\} = 0\end{aligned}$$

To analyze the system of ordinary differential equations above we have the following assignments:

$$\begin{aligned}\dot{h}_4 &= 0 \leadsto h_4 \text{ is a constant function } \leadsto h_4 = k_1, \\ \dot{h}_5 &= 0 \leadsto h_5 \text{ is a constant function } \leadsto h_5 = k_2, \\ \dot{h}_6 &= 0 \leadsto h_6 \text{ is a constant function } \leadsto h_6 = k_3\end{aligned}$$

and the system of linear ordinary differential equations, which can be written in the matrix form as

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{pmatrix} = \begin{pmatrix} 0 & -k_1 & k_2 \\ k_1 & 0 & -k_3 \\ -k_2 & k_3 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

The main tool to analyze the systems of linear ODEs, is eigenvalues and eigenvectors computations. In our case, the straightforward computation

$$\begin{vmatrix} -\lambda & -k_1 & k_2 \\ k_1 & -\lambda & -k_3 \\ -k_2 & k_3 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda(k_1^2 + k_2^2 + k_3^2) = -\lambda(\lambda^2 + (k_1^2 + k_2^2 + k_3^2)) = 0$$

leads to one real and two complex conjugated eigenvalues:

$$\lambda_1 = 0, \quad \lambda_{\pm i} = \pm i\sqrt{k_1^2 + k_2^2 + k_3^2},$$

where the eigenvectors are

$$\begin{aligned}v_0 &:= \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix}, \quad v_i := \begin{pmatrix} k_1^2 + k_2^2 \\ -i\sqrt{k_1^2 + k_2^2 + k_3^2}k_1 + k_2k_3 \\ i\sqrt{k_1^2 + k_2^2 + k_3^2}k_2 - k_1k_3 \end{pmatrix}, \\ v_{-i} &:= \begin{pmatrix} k_1^2 + k_2^2 \\ i\sqrt{k_1^2 + k_2^2 + k_3^2}k_1 - k_2k_3 \\ -i\sqrt{k_1^2 + k_2^2 + k_3^2}k_2 - k_1k_3 \end{pmatrix},\end{aligned}$$

respectively. From the theory of linear systems of ODEs in matrix form, the solution can be described as a linear combination of real and imaginary part of $u = v_o \exp(0t) + v_i \exp(it) + v_{-i} \exp(-it)$. The straightforward computation leads to

$$\begin{aligned}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} v_0^x \\ v_0^y \\ v_0^d \end{pmatrix} + \begin{pmatrix} v_i^x \exp(it) \\ v_i^y \exp(it) \\ v_i^d \exp(it) \end{pmatrix} + \begin{pmatrix} v_{-i}^x \exp(-it) \\ v_{-i}^y \exp(-it) \\ v_{-i}^d \exp(-it) \end{pmatrix} \\
&= \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix} + \begin{pmatrix} (k_1^2 + k_2^2)(\cos(t) + i \sin(t)) \\ (-ikk_1 + k_2k_3)(\cos(t) + i \sin(t)) \\ (ikk_2 - k_1k_3)(\cos(t) + i \sin(t)) \end{pmatrix} \\
&\quad + \begin{pmatrix} (k_1^2 + k_2^2)(\cos(t) - i \sin(t)) \\ (ikk_1 - k_2k_3)(\cos(t) - i \sin(t)) \\ (-ikk_2 - k_1k_3)(\cos(t) - i \sin(t)) \end{pmatrix} \\
&= \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix} + \begin{pmatrix} 2(k_1^2 + k_2^2) \cos(t) \\ 2kk_1 \sin(t) \\ -2k_1k_3 \cos(t) - 2kk_2 \sin(t) \end{pmatrix} + i \begin{pmatrix} 0 \\ 2k_2k_3 \sin(t) \\ 0 \end{pmatrix},
\end{aligned}$$

where $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$.

5 Local controllability

To perform the Lie bracket motions we apply a periodic input, i.e. for the vector fields $X_4 = [X_1, X_2]$, $X_5 = [X_1, X_3]$, $X_6 = [X_2, X_3]$, respectively, the input

$$\begin{aligned}
v_1(t) &= (-A\omega \sin \omega t, A\omega \cos \omega t, 0), \\
v_2(t) &= (0, -A\omega \sin \omega t, A\omega \cos \omega t), \\
v_3(t) &= (-A\omega \sin \omega t, 0, A\omega \cos \omega t)
\end{aligned}$$

is applied, because, according to [6], the Lie bracket of a pair of vector fields corresponds to the direction of a displacement in the state space as a result of a periodic input with sufficiently small amplitude A , i.e. the bracket motions are generated by periodic combination of the vector controlling fields. The theoretic approach above leads to four new control sequences with respect to the parameters k_1 , k_2 and k_3 . These control sequences are the following:

$$\begin{aligned}
e_1(t) &= (2k_1^2 \cos(t), 2kk_1 \sin(t), -2k_1k_3 \cos(t)), \\
e_2(t) &= ((k_1^2 + k_2^2) \cos(t), 2k_1k_3 \sin(t), -2kk_2 \sin(t)), \\
e_{3A}(t) &= (2k_2^2 \cos(t), 0, -2kk_2 \sin(t)), \\
e_{3B}(t) &= (0, 2k_1k_3 \sin(t), 0).
\end{aligned}$$

It is easy to see that by appropriate choice of coordinates k_1, k_2 and k_3 we can get classical periodic inputs as a normal extremal of the underlying sub-Riemannian structure. For example, the choice $k_1 \mapsto -\sqrt{A\omega}, k_2 = k_3 = 0$ and $t \mapsto \omega t$ leads to identification $e_1(t) \cong -v_1(t)$.

References

- [1] A. Agrachev, D. Barilari, U. Boscain, Introduction to Riemannian and sub-Riemannian geometry, Preprint SISSA, 2016
- [2] A. A. Ardentov, Yu. L. Sachkov, Extremal trajectories in a nilpotent sub-Riemannian problem on the Engel group, Russian Academy of Sciences Sbornik Mathematics 202(11):1593, 2012.
- [3] Hrdina, J., Návrát, A., Vašík, P.: Nilpotent approximation of a trident snake robot controlling distribution, arXiv:1607.08500 [math.DG], 2016.
- [4] Hrdina, J., Návrát, A., Vašík, P., Matoušek, R.: *CGA-based robotic snake control*, Adv. Appl. Clifford Algebr. (in print), 1–12 online (2016).
- [5] Ishikawa, M.: Trident snake robot: locomotion analysis and control. NOLCOS, IFAC Symposium on Nonlinear Control Systems 6, 1169–1174 (2004).
- [6] Ishikawa, M., Minami, Y., Sugie, T.: Development and control experiment of the trident snake robot. IEEE/ASME Trans. Mechatronics **15**(1), 9–15 (2010).
- [7] Jean, F.: *Control of Nonholonomic Systems: From Sub-Riemannian Geometry to Motion Planning*, SpringerBriefs in Mathematics, Springer, 2014.
- [8] Murray, R. M., Zexiang, L., Sastry, S. S.: *A Mathematical Introduction to Robotic Manipulation*, CRC Press, 1994.
- [9] Selig, J.M.: *Geometric Fundamentals of Robotics*, Springer, Monographs in Computer Science, 2004.